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Research Article

Series Solution for Single and System of Non-linear Volterra Integral Equations

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ABSTRACT

In this paper, Taylor series expansion is used for finding the approximate series solution of single and system of non-linear Volterra integral equations of the second kind. The method allows us to overcome the difficulty caused by integrals and non-linearity; also, it has more precise and rapidly convergent to the exact solution. Four examples are considered to show the accuracy and efficiency of the method

Keywords: Integral equations, non-linear, Taylor expansion, approximate solution

INTRODUCTION

One of the most useful aspects of numerical analysis is to study approximate or/and numerical solution of mathematical equations, including differential and integral equations.

The integral equation appears in many scientific problems, particularly, engineering problems such as fluid and solid mechanics, for example, potential flow theory, dynamics of population, and elasticity of structures.[1-3]

At the end of the 18^{th} century, the Volterra integral equation (VIE) was introduced by Volterra, in which a variable appears at the upper boundary of the integrals, and it has first and second kind.[4]

There are very few available techniques to handle the solution of integral equation in general, and non-linear integral equation in particular. Therefore, we introduce approximation, because we aim to carry out some mathematical calculation involving these knowing functions such as performing integrations and differentiations. Taylor's expansion is a traditional and very important tools in the approximation theory, in which we can replace certain complicated functions by an approximated and simple polynomial.[5]

Taylor's expansion[6] and its properties are used to find an approximated solution of different types of equations. Some important, efficient, and classical numerical techniques for solving different equations are based on the Taylor series expansion, for instance, Euler method, modified Euler method, and fourth-order Runge–Kutta method.[7-10]

Many works for finding a numerical solution of non-linear VIEs have been proposed (see[11-13]). Kanwel and Liu[14] presented an algebraic technique based on the Taylor expansion technique to find solution of linear integral equation, while Saeed[5] used this expansion to treat system of linear problems, including Volterra and Fredholm integral equations and integrodifferential equations.

In this study, we are concerned with the use of Taylor series expansion to find an approximate series solution of the single and system of non-linear VIEs of the second kind, that is,

$$u(x) = v(x) + \int_a^x k(x,t,u(t))\,dt, \tag{1}$$

where $v(x)$ is known, and it is continuous and differentiable on the given domain (a, b), and, $u(x)$ is undetermined and required function, and $k(x, t, u(t))$ is called kernel function and continuous with respect to the three variables x, t and $u(t)$ on the domain $a \le x \le b$, $a \le t \le x$.

SOLUTION OF EQUATION (1) BY TAYLOR EXPANSION

Expand $u(x)$ in Taylor series expansion at $x=c$, i.e.

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$$u(x) = \sum_{i=0}^{\infty} \frac{1}{i!}(x-a)^i u^{(i)}(a) \quad (2)$$

Hereafter, we will attempt to find $u(a)$, $u'(a)$, $u''(a)$.

When we put $x=a$, in equation (1) we have $u(a)=v(a)$. Take the first derivative for equation (1) with respect to x using generalized Leibnitz formula[9] and get the different equations such as first-order approximation which is known as Euler's method, second-order approximation which is called predictor-corrector or modified Euler method and fourth-order approximation which is called fourth Runge–Kutta scheme.[7-10]

In this paper, we were concerned to use the Taylor series expansion (2) to find an approximate solution of the non-linear VIE of the second kind. Therefore

$$u'(x) = v'(x) + \int_a^x k'(x,t,u(t))\,dt + k(x,x,u(x)), \quad (3)$$

when we put $x=a$, in this equation we have

$$u'(a) = v'(a) + k(a,a,u(a)).$$

Again, to find $u''(a)$, differentiate equation (3) with respect to x and get

$$u''(x) = v''(x) + \int_a^x k''(x,t,u(t))\,dt + 2k'(x,x,u(x)) \quad (4)$$

Again, when setting $x=a$, in equation (4) we get

$$u''(a) = v''(a) + 2k'(a,a,u(a)),$$

and so on we repeat this process n-time to obtain

$$u^{(n)}(a) = u^{(n)}(a) + nk^{(n-1)}(a,a,u(a)),\ n=1, 2, \ldots \quad (5)$$

Putting the values of equation (5) for $n=1, 2, \ldots$ in the Taylor series expansion (2) in the order, we can obtain the finite and approximate series solution of equation (1).

NUMERICAL EXAMPLES

Example 1

Consider the problem

$$u(x) = 1 - x + \int_a^x (x\,e^{t(x-2t)} + e^{-2t^2})(u(t))^2\ dt$$

Where the exact solution is $u(x) = e^{x^2}$.

Solution:

By using the procedure in section two, where $a=0$, we obtained

$$u(0)=1,\ u^{(2)}(0)=2,\ u^{(4)}(0)=12,\ u^{(6)}(0)=120, u^{(8)}(0)=1680,$$

and

$$u^{(1)}(0) = u^{(3)}(0) = u^{(5)}(0) = u^{(7)}(0) = \ldots = 0.$$

Using all these values in the Taylor series expansion (2), we arrive

$$u(x) \approx 1 + x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{6} + \frac{(x^2)^4}{24} + \cdots + \frac{(x^2)^n}{n}$$

This equation is the n^{th} partial sum of the Maclaurin series of e^{x^2}.

Example 2

Consider the non-linear VIE

$$u(x) = \int_0^x \frac{1+u(t)}{1+t}\, dt$$

Where the actual solution is $u(x)=x$ and $v(x)=0$.

Solution:

From equation (5), when $x=0$ we obtain

$$u(0)=0,\ u'(0)=1,\ u^{(2)}(0)=0, u^{(3)}(0)=0, \ldots u^{(n)}(0)=0$$

Using all these obtained values in the Taylor series expansion in (2) to arrive, the exact solution $u(x)=x$.

This approach also can be exploited for solving system of non-linear VIE by using Lebibntz Generalized Formula[15] which state that:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} K(x,t,u(t))dt = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} K(x,t,u(t))dt + K(x,b(x),u(b(x)))\frac{db(x)}{dx} - K(x,a(x),f(a(x)))\frac{da(x)}{dx} \quad (6)$$

In this investigation $a(x)=a$ is constant and $b(x)=x$, therefore (6) reduce to

$$\frac{d}{dx}\int_{a(x)}^{b(x)} K(x,t,u(t))dt = \int_{b(x)}^{a(x)} \frac{\partial}{\partial x} K(x,t,u(t))dt + K(x,x,u(x)) \quad (7)$$

We can use this principle to convert initial or boundary value problem into an integral equation. For instance, suppose we the following initial value problem

$$u''(x) = K(x,u(x),u'(x)) \quad (8)$$

with $u(a)=u_0$ and $u'(a)=u_1$ are given. Take integral for both side of (8), we get

$$\int_{t=a}^{t=x} u''(t)\,dt = \int_{t=a}^{t=x} K(t,u(t),u'(t))\,dt \quad (9)$$

use principle of (7) for both side of (9), we obtain

$$u'(x) - u_1 = \int_{t=a}^{t=x} K(t,u(t),u'(t))\,dt \quad (10)$$

Take integral again for both side of (10), we can rewrite (10) as,

$$\int_a^x (u'(t) - u_1)\,dt = \int_a^x\int_a^z K(z,u(z),u'(z))\,dzdx \quad (11)$$

$$u(x) - u(a) - u_1(x-a) = \int_a^x (x-a)K(z,u(z),u'(z))\,dz$$

Thus, we get,

$$u(x)=u(a)+u_1(x-a)+\int_a^x (x-a)K(z,u(z),u'(z))\,dz$$

$$=v(x)+\int_a^x (x-a)K(t,u(t),u'(t))\,dt$$

which is non-linear of the second kind and $v(x)= v(a)+ u_1(x-a)$.

In addition, equation (7) can be used for each equation in the system of non-linear VIEs as illustrated in the following two examples.

Example 3[16]

Consider the following system

$$u_1(x)=\frac{1}{4}-\frac{1}{4}e^{2x}+\int_0^x (x-t)u_2^2(t)\,dt$$

$$u_2(x)=-xe^{-x}+2e^x-1+\int_0^x t\,e^{-2u_1(t)}\,dt$$

Where actual solution $u_1(x)=-\frac{1}{2}x$ and $u_2(x)=e^x$.

$u_1(0)=1$ and $u_2(0)=1$

$$u_1'(x)=-\frac{1}{2}e^{2x}+\int_0^x \frac{\partial}{\partial x}\left((x-t)u_2^2(t)\right)dt+(x-x)u_2^2(x)$$

$$u_2'(x)=-e^{-x}+xe^{-x}+2e^x+\int_0^x \frac{\partial}{\partial x}\left(t\,e^{-2u_1(t)}\right)dt+x\,e^{-2u_1(x)}$$

So, $u_1'(0)=-\frac{1}{2}$ and $u_2'(0)=1$

$u_1''(0)=0$ and $u_2''(0)=1.$ by continuing in this process, we can the exact solution.

Example 4

Consider the following system

$$u_1(x)=\sec(x)-x+\int_0^x u_1^2(t)-u_2^2(t)\,dt$$

$$u_2(x)=3\tan(x)-x-\int_0^x u_1^2(t)+u_2^2(t)\,dt$$

With exact solution $u_1(x)=\sec(x)$ and $u_2(x)=\tan(x)$.

$u_1(0)=1$ and $u_2(0)=1$

$$u_1'(x)=\sec(x)\ \tan(x)-1+\int_0^x \frac{\partial}{\partial x}\left(u_1^2(t)-u_2^2(t)\right)dt+\left(u_1^2(x)-u_2^2(x)\right)$$

$$u_2'(x)=3\ \sec^2(x)-1-\int_0^x \frac{\partial}{\partial x}\left(u_1^2(t)+u_2^2(t)\right)dt-\left(u_1^2(x)+u_2^2(x)\right)$$

So, $u_1'(0)=0$ and $u_2'(0)=1$ by the same manner, we can get the exact solution.

CONCLUSION

Taylor series expansion was applied successfully to find an approximate or exact series solution of single and system of non-linear VIEs in the second kind. The method was tested by taking four numerical examples and good results were obtained. We also conclude that the solution obtained by Taylor series expansion is given by a function and not only at some points and numerical computations of this method are simple and the convergence is satisfactory.

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