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RESEARCH ARTICLE

On Generalized Recurrent Finsler Spaces of Higher Order with Berwald's Curvature Tensor

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ABSTRACT

In this article, we presented the recurrent of higher order in Finsler space F_n for projective tensor K^i_{jkh} which satisfies a generalized five recurrence with respect to Berwald's connection parameters G^i_{kh}, and we have some theorems and some identities also we get more results in a recurrent and generalized 5-recurrent F_n by using the sense of Berwald's curvature tensor.

Keywords: Generalized $\mathcal{B}K$ -five-recurrent, Berwald's derivative of fifth order, curvature tensor K^i_{jkh}, curvature tensor R^i_{jkh}, Weyl projective curvature tensor W^i_{jkh}

INTRODUCTION

The generalized recurrent Finsler space used the sense of Berwald curvature tensor discussed by Abdallah,[1] AL-Qashbari and Qasem,[2] and some others. A n-dimensional Riemannian space of recurrent was introduced and studied by Rund[3]. Some properties for Weyl's curvature tensor studied by Al-Qashbari[4],[15],[17],[19],[20],[21],[22],[23],[24], Ahsan and Ali[5], Abu-Donia *et al.*,[6] Emamian *et al*.[7],[13],[14],[16],[18] and Qasem *et al*.[8],[25],[26] The generalized birecurrent, trirecurrent Finsler space, and higher order recurrent are studied. Furthermore, Awed[9] introduced the curvature tensors for the space-time of general relativity. The decomposability of certain generalized $\mathcal{B}K$ -recurrent Finsler space have been studied by Baleedi[10], Al-Qufail,[11] and Pandey *et al*.[12] and others.

Berwald's covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor filed T^i_j with respect to x^k is given by

$$\mathcal{B}_k T^i_j = \partial_k T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk} \tag{1.1}$$

Berwald's covariant derivative of the metric function and the vector y^i vanish identically, i.e.

$$\text{(a) } \mathcal{B}_k F = 0 \text{ and (b) } \mathcal{B}_k y^j = 0. \tag{1.2}$$

But Berwald's covariant derivative of the metric tensor g_{ij} does not vanish and is given by

$$\mathcal{B}_k g_{ij} = -2C_{ijklh} y^h = -2y^h \mathcal{B}_h C_{ijk} \tag{1.3}$$

The vectors y_i and y^i satisfy the following relations

$$\text{(a) } g_{ij} y^j = y_i, \text{ (b) } y_i y^i = F^2, \text{ (c) } \delta^k_j y^j = y^k \text{ and (d) } \delta^k_h g_{ik} = g_{ih} \tag{1.4}$$

The two sets of quantities g_{ij} and its associate tensor g^{ij} are related by[11]

$$g_{ij} g^{jk} = \delta^k_i = \begin{cases} 1 & , \quad if \quad i = k \\ 0 & , \quad if \quad i \neq k \end{cases} \tag{1.5}$$

The tensor C_{ijk} defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{jk} = \frac{1}{4}\dot{\partial}_i \dot{\partial}_j \dot{\partial}_k F^2 \tag{1.6}$$

is known as (h) hv-torsion tensor.

The torsion tensor C^h_{ik} and its associate torsion tensor C_{ijk} are related by

$$\text{(a) } C^i_{jk} y^j = C^i_{kj} y^j = 0, \text{ (b) } C_{ijk} y^i = C_{ijk} y^j = C_{ijk} y^k = 0 \text{ and (c) } \delta^k_h C_{ijk} = C_{hij} \tag{1.7}$$

The tensor K^i_{rkj} as defined above is called Cartan's fourth

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curvature tensor, this tensor is positively homogeneous of degree zero in the directional argument

(a) $K^i_{rkj} = \partial_j \Gamma^{*i}_{kr} + (\dot{\partial}_l \Gamma^{*i}_{rj}) G^l_k + \Gamma^{*i}_{mj}\Gamma^{*m}_{kr} - \partial_k \Gamma^{*i}_{jr} + (\dot{\partial}_l \Gamma^{*i}_{rk}) G^l_j + \Gamma^{*i}_{mk}\Gamma^{*m}_{jr}$
and (b) $K^i_{jkh} = -K^i_{jhk}$ (1.8)

The tensor K^i_{jkh} satisfies the relation too.

(a) $K^i_{jkh} y^j = H^i_{kh}$ and (b) $H^i_{jkh} = K^i_{jkh} + y^m (\dot{\partial}_j K^i_{mkh})$ (1.9)

The curvature tensor K^i_{jkh} satisfies the following identities known as Bianchi identities[27]

$$K^i_{jkh} + K^i_{hjk} + K^i_{khj} = 0 \quad (1.10)$$

Ricci tensor K_{jk}, the curvature vector K_j and the curvature scalar K are given by:

(a) $K^i_{jki} = K_{jk}$, (b) $K_{jk} y^k = K_j$ and (c) $K_{jk} y^j = H_k$ (1.11)

The quantities H^i_{jkh} and H^i_{kh} form the components of tensors and they called h-curvature tensor of Berwald and torsion tensor respectively are defined as

(a) $H^i_{jkh} = \partial_j G^i_{kh} + G^r_{kh} G^i_{rj} + G^i_{rhj} G^r_k \; - h/k * H^i_{jkh} \; - h/k$:

(b) $H^i_{kh} = \partial_h G^i_k + G^r_k C^i_{rh} - h/k$ (1.12)

They are also related by:

(a) $H^i_{jkh} y^j = H^i_{kh}$ and (b) $H^i_{jkh} = \partial_j H^i_{kh}$ (1.13)

The tensor H^i_h , called the deviation tensor, given by:

$$H^i_h = 2\partial_h G^i - \partial_r G^i_h y^r + 2G^i_{hs} G^s - G^i_s G^s_h \quad (1.14)$$

The curvature tensor R^i_{jkh}, Ricci tensor R_{jk}, tensor H_{kh}, curvature vector H_k and scalar curvature H are connected by the following:

(a) $H^i_{jk} y^j = H^i_k$, (b) $H_{jk} = H^r_{jkr}$, (c) $H_j = H^i_{ji}$, (d) $R^i_{jkh} y^j = H^i_{kh}$,

(e) $R_{jk} y^j = H_k$, (f)., (j) $R^i_i = R$ and (h) $R^i_{jki} = R_{jk}$ (1.15)

also connected by:

(a) $H_{kh} = \dot{\partial}_k H_h$, (b) $H_{kh} y^k = H_h$ and (d) $H_k y^k = H^k_k = (n-1)H$ (1.16)

The Weyl curvature tensor denoted as W^i_{jkh} is defined by

$$W^i_{jkh} = H^i_{jkh} + \frac{\delta^i_j}{(n+1)}\left(H_{kh} - H_{hk}\right) + \frac{\dot{x}^i}{(n+1)}\left(\dot{\partial}_j H_{kh} - \dot{\partial}_j H_{hk}\right) + \left\{ \frac{\delta^i_k}{(n^2-1)}\left(n H_{jh} + H_{hj} + \dot{x}^r \dot{\partial}_j H_{hr}\right) - \frac{\delta^i_h}{(n^2-1)} \left\{\left(n H_{jk} + H_{kj} + \dot{x}^r \dot{\partial}_j H_{kr}\right)\right\} \right\} \quad (1.17)$$

The tensors W^i_{jkh}, W^i_{jk} and W_{jk} satisfies the following identities.

(a) $W^i_{jkh} y^j = W^i_{kh}$, (b) $W^i_{jk} y^j = W^i_k$ and (c) $W^i_{jki} = W_{jk}$ (1.18)

Notations: K^i_{jkh} : Cartan's 4th Curvature Tensor; R^i_{jkh} : Cartan's 3th Curvature Tensor;

H^i_{jkh} : Berwald Curvature Tensor; K_{kh}: K-Ricci Tensor, K_k: Curvature Vector and K: Scalar Curvature.

ON GENERALIZED –*BK*-FIVE-RECURRENT SPACE

In this, proposal is defined as $\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l$ is derivative of fifth order, for projective curvature tensor K^i_{jkh} which is defined as:

$$\begin{aligned}
\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K^i_{jkh} &= \lambda_{lmnsr} K^i_{jkh} + \mu_{lmnsr}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) \\
&-2\ \mu_{lmns}\mathcal{B}_q y^q \left(\delta^i_h C_{jkr} - \delta^i_k C_{jhr}\right) - 2\ \mu_{lmnr}\mathcal{B}_q y^q \left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right) \\
&-2\mu_{lmn}\mathcal{B}_r\mathcal{B}_q y^q \left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right) - 2\mu_{lmsr}\mathcal{B}_q y^q \left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) \\
&-2\ \mu_{lms}\mathcal{B}_r\mathcal{B}_q y^q \left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\ \mu_{lmr}\mathcal{B}_s\mathcal{B}_q y^q \left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) \\
&-2\ \mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q \left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\mu_{lnsr}\mathcal{B}_p y^q \left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) \\
&-2\mu_{lns}\mathcal{B}_r\mathcal{B}_p y^q \left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\ \mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q \left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) \\
&-2\ \mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q \left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\ \mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q \left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) \\
&-2\ \mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q \left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\ \mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q \left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) \\
&-2\ \mu_l\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q \left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right).
\end{aligned} \quad (2.1)$$

Where $K^i_{jkh} \neq 0$ and $\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l$ is derivative of five order with respect to x^l, x^m, x^n, x, and x_r, respectively, the quantities λ_{lmnsr} and μ_{lmnsr} are non-null covariant vectors field.

Result 2.1. Every generalized $\mathcal{B}K$ -recurrent is generalized $\mathcal{B}K$ -five recurrent.

Definition 2.1. A Finsler space for the tensor K^i_{jkh} is known to satisfy the condition (2.1), and will be called generalized-five recurrent space. We shall call such Finsler space a generalized $\mathcal{B}K$ -five-recurrent space and is denoted by $G\mathcal{B}K$ -$FIRF_n$.

Transvecting (2. 1) by y^j, using (1.2b), (1.9a), (1.4a) and (1.7b), we get

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^i_{kh} = a_{lmnsr} H^i_{kh} + b_{lmnsr}\left(\delta^i_h y_k - \delta^i_k y_h\right) \quad (2.2)$$

Transvecting (2.2) by y^k, using (1.2b), (1.15a), (1.4b) and (1.4c), we get

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^i_h = a_{lmnsr} H^i_h + b_{lmnsr}\left(\delta^i_h F^2 - y_h y^i\right) \quad (2.3)$$

Thus, we conclude

Theorem 2.1. In G$\mathcal{B}$K -FIRF$_n$, Berwald derivative of the five orders for torsion tensor H^i_{kh} and the deviation tensor H^i_h are given by the conditions (2.2) and (2.3), respectively.

Putting $i=h$ in (2.2) and (2.3), using (1.4b), (1.4c), (1.5), (1.15b), and (1.15c), we get

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H_k = a_{lmnsr} H_k + (n-1) b_{lmnsr} y_k \text{ and} \quad (2.4)$$

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H = a_{lmnsr} H + (n-1) b_{lmnsr} F^2 \quad (2.5)$$

The conditions (2.4) and (2.5), show that the curvature vector H_k and the curvature scalar H cannot vanish because one of these would imply $a_{lmnsr}=0$ and $b_{lmnsr}=0$, that is a contradiction.

Thus, we conclude,

Theorem 2.2. In $G\mathcal{B}K$-FIRF$_n$ the curvature vector W_K and the curvature scalar W are given by the conditions (2.4) and (2.5), respectively, are non-vanishing.

Contracting the indices i and h in (2.1), using (1.11a), (1.4c) and (1.7c), we get

$$\begin{aligned}
&\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K_{jk} = \lambda_{lmnsr} K_{jk} + (n-1)\mu_{lmnsr}\, g_{jk} - 2(n-1)\\
&\mu_{lmns}\mathcal{B}_q y^q C_{jkr}\\
&-2(n-1)\ \mu_{lmnr}\mathcal{B}_q y^q C_{jks} - 2(n-1)\mu_{lmn}\mathcal{B}_r\mathcal{B}_q y^q C_{jks}\\
&-2(n-1)\mu_{lmsr}\mathcal{B}_q y^q C_{jkn}\\
&-2(n-1)\mu_{lms}\mathcal{B}_r\mathcal{B}_q y^q C_{jkn} - 2(n-1)\mu_{lmr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkn} \qquad (2.6)\\
&-2(n-1)\mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkn}\\
&-2(n-1)\mu_{lnsr}\mathcal{B}_p y^q C_{jkm} - 2(n-1)\mu_{lns}\mathcal{B}_r\mathcal{B}_p y^q C_{jkm}\\
&-2(n-1)\mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkm}\\
&-2(n-1)\mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkm} - 2(n-1)\mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}\\
&-2(n-1)\mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}\\
&-2(n-1)\mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q\, y^q C_{jkm} - 2(n-1)\ \ \mu_l\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}
\end{aligned}$$

Thus, we conclude

Theorem 2.3. In $G\mathcal{B}K$-FIRF$_n$, Berwald derivative of the five orders for Ricci tensor K_{jk} is given by the condition (2.6).

Transvecting (2.6) by y^j, using (1.2b), (1.11b), (1.4a), and (1.7b), we get

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K_j = \lambda_{lmnsr} K_j + (n-1)\mu_{lmnsr}\, y_j \qquad (2.7)$$

Thus, we get

Theorem 2.4. In $G\mathcal{B}K$-FIRF$_n$, Berwald derivative of the five orders for the curvature vector K_j is given by the condition (2.7).

DIVERGENCE OF *K*-TENSOR AND OTHER CURVATURE TENSORS

In this section, we will obtain the necessary and sufficient conditions for tensors to be interpreted to generalized recurrent in $G\mathcal{B}K$-$FIRF_n$. It is known that curvature tensor R^i_{jkh} and curvature tensor K^i_{jkh} are connected by the formula[27]

$$R^i_{jkh} = K^i_{jkh} + C^i_{jp} H^p_{hk}\ . \qquad (3.1)$$

Taking derivative of 5th order of (3.1), with respect to x^l, x^m, x^n, x^s, and x^r, successively, we get:

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R^i_{jkh} = \mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K^i_{jkh} + \mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{jp} H^p_{hk}\right) \qquad (3.2)$$

Using the condition (2.1) in (3.2), we get:

$$\begin{aligned}
&\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R^i_{jkh} = \lambda_{lmnsr} K^i_{jkh} + \mu_{lmnsr}\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right)\\
&-2\ \mu_{lmns}\mathcal{B}_q y^q\left(\delta^i_h C_{jkr} - \delta^i_k C_{jhr}\right) - 2\ \mu_{lmnr}\mathcal{B}_q y^q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right)\\
&-2\mu_{lmn}\mathcal{B}_r\mathcal{B}_q\, y^q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right) - 2\mu_{lmsr}\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right)\\
&-2\ \mu_{lms}\mathcal{B}_r\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\ \mu_{lmr}\mathcal{B}_s\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right)\\
&-2\ \mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\mu_{lnsr}\mathcal{B}_p\, y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) \qquad (3.3)\\
&-2\mu_{lns}\mathcal{B}_r\mathcal{B}_p\, y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\ \mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)\\
&-2\ \mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\ \mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)\\
&-2\ \mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\ \mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)\\
&-2\ \mu_l\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\right) + \mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{jp} H^p_{hk}\right)
\end{aligned}$$

By using (3.1), the above equation can be written as:

$$\begin{aligned}
&\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R^i_{jkh} = a_{lmnsr} R^i_{jkh} - a_{lmnsr} C^i_{jp} H^p_{hk} + b_{lmns}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)\\
&-2\ \mu_{lmns}\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkr} - \delta^i_k C_{jhr}\right) - 2\ \mu_{lmnr}\mathcal{B}_q\, y^q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right)\\
&-2\mu_{lmn}\mathcal{B}_r\mathcal{B}_q\, y^q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right) - 2\mu_{lmsr}\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right)\\
&-2\ \mu_{lms}\mathcal{B}_r\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\ \mu_{lmr}\mathcal{B}_s\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right)\\
&-2\ \mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\mu_{lnsr}\mathcal{B}_p\, y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)\\
&-2\mu_{lns}\mathcal{B}_r\mathcal{B}_p y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\ \mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) \qquad (3.4)\\
&-2\ \mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\ \mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)\\
&-2\ \mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\ \mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)\\
&-2\ \mu_l\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\right) + \mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{jp} H^p_{hk}\right).
\end{aligned}$$

This shows that,

$$\begin{aligned}
&\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R^i_{jkh} = a_{lmnsr} R^i_{jkh} + b_{lmns}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)\\
&-2\ \mu_{lmns}\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkr} - \delta^i_k C_{jhr}\right) - 2\ \mu_{lmnr}\mathcal{B}_q\, y^q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right)\\
&-2\mu_{lmn}\mathcal{B}_r\mathcal{B}_q\, y^q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right) - 2\mu_{lmsr}\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right)\\
&-2\ \mu_{lms}\mathcal{B}_r\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\ \mu_{lmr}\mathcal{B}_s\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right)\\
&-2\ \mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q\, y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\mu_{lnsr}\mathcal{B}_p y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) \qquad (3.5)
\end{aligned}$$

$$-2\mu_{lns}\mathcal{B}_r\mathcal{B}_p y^q\left(\delta^i_h C_{jkm}-\delta^i_k C_{jhm}\right)-2\ \mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkm}-\delta^i_k C_{jhm}\right)$$

$$-2\ \mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkm}-\delta^i_k C_{jhm}\right)-2\ \mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm}-\delta^i_k C_{jhm}\right)$$

$$-2\ \mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm}-\delta^i_k C_{jhm}\right)-2\ \mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm}-\delta^i_k C_{jhm}\right)$$

$$-2\ \mu_{l}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm}-\delta^i_k C_{jhm}\right).$$

If and only if

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{jp}H^p_{hk}\right)=a_{lmnsr}\left(C^i_{jp}H^p_{hk}\right) \tag{3.6}$$

Thus, we conclude

Theorem 3.1 In $G\mathcal{B}K$-$FIRF_n$, Cartan's fourth curvature tensor R^i_{jkh} is generalized five recurrent Finsler space if and only if the condition $\left(C^i_{jp}H^p_{hk}\right)$ is five recurrent Finsler space.

Transvecting (3.4) by y^j, using (1.2b), (1.15d), (1.4a), (1.7a) and (1.7b), we get

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^i_{kh}=a_{lmnsr}H^i_{kh}+b_{lmns}\left(\delta^i_h y_k-\delta^i_k y_h\right) \tag{3.7}$$

The following is derived.

Theorem 3.2. In $G\mathcal{B}K$-FIRF$_n$, Berwald derivative of the five orders for the torsion tensor H^i_{kh} is generalized five recurrent Finsler space.

Contracting the indices i and h in (3.4), using (1.15h), (1.5), (1.4b), (1.4c), and (1.7c), we get:

$$\begin{aligned}
&\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R_{jk}=a_{lmnsr}R_{jk}-a_{lmnsr}\left(C^i_{jp}H^p_{ik}\right)+(n-1)b_{lmns}g_{jk}\\
&-2(n-1)\ \mu_{lmns}\mathcal{B}_q y^q C_{jkr}-2(n-1)\ \mu_{lmnr}\mathcal{B}_q y^q C_{jks}\\
&-2(n-1)\mu_{lmn}\mathcal{B}_r\mathcal{B}_q y^q C_{jks}\\
&-2(n-1)\mu_{lmsr}\mathcal{B}_q y^q C_{jkn}-2(n-1)\ \mu_{lms}\mathcal{B}_r\mathcal{B}_q y^q C_{jkn}\\
&-2(n-1)\ \mu_{lmr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkn}\\
&-2(n-1)\mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkn}-2(n-1)\mu_{lnsr}\mathcal{B}_p y^q C_{jkm}\\
&-2(n-1)\mu_{lns}\mathcal{B}_r\mathcal{B}_p y^q C_{jkm}\\
&-2(n-1)\mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkm}-2(n-1)\mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkm}\\
&-2(n-1)\mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}\\
&-2(n-1)\mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}-2(n-1)\mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}\\
&-2(n-1)\ \mu_{l}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}+\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{jp}H^p_{ik}\right).
\end{aligned} \tag{3.8}$$

This shows that

$$\begin{aligned}
&\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R_{jk}=a_{lmnsr}R_{jk}+(n-1)b_{lmns}g_{jk}\\
&-2(n-1)\ \mu_{lmns}\mathcal{B}_q y^q C_{jkr}\\
&-2(n-1)\mu_{lmnr}\mathcal{B}_q y^q C_{jks}-2(n-1)\mu_{lmn}\mathcal{B}_r\mathcal{B}_q y^q C_{jks}\\
&-2(n-1)\mu_{lmsr}\mathcal{B}_q y^q C_{jkn}\\
&-2(n-1)\mu_{lms}\mathcal{B}_r\mathcal{B}_q y^q C_{jkn}-2(n-1)\mu_{lmr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkn}\\
&-2(n-1)\mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkn}\\
&-2(n-1)\mu_{lnsr}\mathcal{B}_p y^q C_{jkm}-2(n-1)\mu_{lns}\mathcal{B}_r\mathcal{B}_p y^q C_{jkm}\\
&-2(n-1)\mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkm}\\
&-2(n-1)\mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkm}-2(n-1)\mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}\\
&-2(n-1)\mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}\\
&-2(n-1)\ \mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}-2(n-1)\ \mu_{l}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q\ \ C_{jkm}.
\end{aligned} \tag{3.9}$$

If and only if

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{jp}H^p_{ik}\right)=a_{lmnsr}\left(C^i_{jp}H^p_{ik}\right) \tag{3.10}$$

This following is derived

Theorem 3.3. In $G\mathcal{B}K$-FIRF$_n$, R-Ricci tensor R_{jk} is generalized five recurrent Finsler space if and only if the condition $\left(C^i_{jp}H^p_{ik}\right)$ is five recurrent Finsler space.

For a Riemannian space V_4, when $n=4$, the equation (3.8), shows that

$$\begin{aligned}
&\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R_{jk}=a_{lmnsr}R_{jk}+3b_{lmns}g_{jk}-6\ \mu_{lmns}\mathcal{B}_q y^q C_{jkr}\\
&-6\ \mu_{lmnr}\mathcal{B}_q y^q C_{jks}\\
&-6\mu_{lmn}\mathcal{B}_r\mathcal{B}_q y^q C_{jks}-6\mu_{lmsr}\mathcal{B}_q y^q C_{jkn}-6\ \mu_{lms}\mathcal{B}_r\mathcal{B}_q y^q C_{jkn}\\
&-6\ \mu_{lmr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkn}\\
&-6\ \mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkn}-6\mu_{lnsr}\mathcal{B}_p y^q C_{jkm}-6\mu_{lns}\mathcal{B}_r\mathcal{B}_p y^q C_{jkm}\\
&-6\ \mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkm}\\
&-6\ \mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkm}-6\ \mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}-6\ \mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}\\
&-6\ \mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}-6\ \mu_{l}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}
\end{aligned} \tag{3.11}$$

If and only if

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{jp}H^p_{ik}\right)=a_{lmnsr}\left(C^i_{jp}H^p_{ik}\right) \tag{3.12}$$

The following is derived.

Theorem 3.4. In $G\mathcal{B}K$-FIRF$_n$, For a Riemannian space V_4, when $n=4$, the R-Ricci tensor R_{jk} is given by condition (3.11) if and only if the condition $\left(C^i_{jp}H^p_{ik}\right)$ is five recurrent Finsler space.

Transvecting (3.9) by y^k, using (1.2b), (1.15f), (1.4a) and (1.7b), we get

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R_j=a_{lmnsr}R_j+(n-1)b_{lmns}y_j \tag{3.13}$$

Thus, we conclude

Theorem 3.5. In $G\mathcal{B}K$-FIRF$_n$, Berwald derivative of the five order for curvature vector R_j is generalized five recurrent Finsler space.

COMPOUND DERIVATIONS OF TENSOR K^i_{jkh}

In this section, we present the relation between the curvature tensor K^i_{jkh} and Wely's projective tensor W^i_{jkh}. It is known that Cartan's third curvature tensor R^i_{jkh} and Wely's projective tensor W^i_{jkh} are connected by the formula[1]

$$W^i_{jkh} = R^i_{jkh} + \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) \tag{4.1}$$

Using (3.1) in (4.1), we get,

$$W^i_{jkh} = K^i_{jkh} + C^i_{jp} H^p_{hk} + \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) \tag{4.2}$$

Taking derivative of 5th order of (4.2), with respect to x^l, x^m, x^n, x^s, and x^r, successively, we get

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_{jkh} = \mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K^i_{jkh} + \mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{jp}H^p_{hk}\right)$$

$$+\frac{1}{3}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right) \tag{4.3}$$

Using (2.1) and substituting the condition (4.2) in (4.3), we get

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_{jkh} = \lambda_{lmnsr} W^i_{jkh} - \lambda_{lmnsr}\left(C^i_{jp}H^p_{hk}\right) - \frac{1}{3}\lambda_{lmnsr}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right)$$

$$+\mu_{lmnsr}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) - 2\mu_{lmns}\mathcal{B}_q y^q\left(\delta^i_h C_{jkr} - \delta^i_k C_{jhr}\right)$$

$$-2\mu_{lmnr}\mathcal{B}_q y^q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right) - 2\mu_{lmn}\mathcal{B}_r\mathcal{B}_q y^q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right)$$

$$-2\mu_{lmsr}\mathcal{B}_q y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\mu_{lms}\mathcal{B}_r\mathcal{B}_q y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right)$$

$$-2\mu_{lmr}\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right)$$

$$-2\mu_{lnsr}\mathcal{B}_p y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\mu_{lns}\mathcal{B}_r\mathcal{B}_p y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)$$

$$-2\mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)$$

$$-2\mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)$$

$$-2\mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\mu_l\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)$$

$$+\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{jp}H^p_{hk}\right) + \frac{1}{3}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right) \tag{4.4}$$

This shows that

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_{jkh} = \lambda_{lmnsr} W^i_{jkh} + \mu_{lmnsr}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$

$$-2\mu_{lmns}\mathcal{B}_q y^q\left(\delta^i_h C_{jkr} - \delta^i_k C_{jhr}\right) - 2\mu_{lmnr}\mathcal{B}_q y^q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right)$$

$$-2\mu_{lmn}\mathcal{B}_r\mathcal{B}_q y^q\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right) - 2\mu_{lmsr}\mathcal{B}_q y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right)$$

$$-2\mu_{lms}\mathcal{B}_r\mathcal{B}_q y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\mu_{lmr}\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) \tag{4.5}$$

$$-2\mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2\mu_{lnsr}\mathcal{B}_p y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)$$

$$-2\mu_{lns}\mathcal{B}_r\mathcal{B}_p y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)$$

$$-2\mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)$$

$$-2\mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)$$

$$-2\mu_l\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)$$

If and only if

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{jp}H^p_{hk}\right) = \lambda_{lmnsr}\left(C^i_{jp}H^p_{hk}\right) \tag{4.6}$$

and

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right) = \lambda_{lmnsr}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right) \tag{4.7}$$

The following is derived

Theorem 4.1. In $G\mathcal{B}K$-FIRF$_n$, Wely's projective tensor W^i_{jkh} is generalized five recurrent Finsler space if and only if the tensors $\left(C^i_{jp}H^p_{hk}\right)$ and $\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right)$ are five recurrent Finsler space.

Transvecting (4.4) by y_j, using (1.2b), (1.18a), (1.4a),(1.7a), (1.15e) and (1.7b), we get

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_{kh} = \lambda_{lmnsr} W^i_{kh} - \frac{1}{3}\lambda_{lmnsr}\left(\delta^i_k H_h - y_k R^i_h\right)$$

$$+\mu_{lmnsr}\left(\delta^i_h y_k - \delta^i_k y_h\right) + \frac{1}{3}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(\delta^i_k H_h - y_k R^i_h\right) \tag{4.8}$$

This shows that

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_{kh} = \lambda_{lmnsr} W^i_{kh} + \mu_{lmnsr}\left(\delta^i_h y_k - \delta^i_k y_h\right) \tag{4.9}$$

If and only if

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(\delta^i_k H_h - y_k R^i_h\right) = \lambda_{lmnsr}\left(\delta^i_k H_h - y_k R^i_h\right) \tag{4.10}$$

This following is derived.

Theorem 4.2. In $G\mathcal{B}K$-FIRF$_n$, the torsion curvature tensor W^i_{kh} is generalized five recurrent Finsler space if and only if the tensor $\left(\delta^i_k H_h - y_k R^i_h\right)$ is five recurrent Finsler space.

Transvecting (4.5) by y^k, using (1.2b), (1.18b), (1.4c), and (1.4b), we get

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_h = \lambda_{lmnsr} W^i_h - \frac{1}{3}\lambda_{lmnsr}\left(y^i H_h - F^2 R^i_h\right)$$

$$+\mu_{lmnsr}\left(\delta_h^i F^2 - y^i y_h\right) + \frac{1}{3}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(y^i H_h - F^2 R_h^i\right) \tag{4.11}$$

This shows that

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W_h^i = \lambda_{lmnsr} W_h^i + \mu_{lmnsr}\left(\delta_h^i F^2 - y^i y_h\right) \tag{4.12}$$

If and only if

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(y^i H_h - F^2 R_h^i\right) = \lambda_{lmnsr}\left(y^i H_h - F^2 R_h^i\right) \tag{4.13}$$

This following is derived

Theorem 4.3. In G$\mathcal{B}$K -FIRF$_n$, the curvature tensor W_h^i is generalized five recurrent Finsler space if and only if the tensor $\left(y^i H_h - F^2 R_h^i\right)$ is five recurrent Finsler space.

Contracting the indices i and h in (4.4), using (1.18c), (1.5), (1.4b), (1.4c), and (1.15j), we get

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W_{jk} = a_{lmnsr} W_{jk} - a_{lmnsr}\left(C_{jp}^i H_{ik}^p\right) \tag{4.14}$$

$$-\frac{1}{3}\lambda_{lmnsr}\left(R_{jk} - g_{jk}R\right)$$

$$+(n-1)b_{lmns}g_{jk} - 2(n-1)\ \mu_{lmns}\mathcal{B}_q y^q C_{jkr} - 2(n-1)\ \mu_{lmnr}\mathcal{B}_q y^q C_{jks}$$

$$-2(n-1)\mu_{lmn}\mathcal{B}_r\mathcal{B}_q y^q C_{jks} - 2(n-1)\mu_{lmsr}\mathcal{B}_q y^q C_{jkn}$$

$$-2(n-1)\ \mu_{lms}\mathcal{B}_r\mathcal{B}_q y^q C_{jkn}$$

$$-2(n-1)\ \mu_{lmr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkn} - 2(n-1)\mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkn}$$

$$-2(n-1)\mu_{lnsr}\mathcal{B}_p y^q C_{jkm}$$

$$-2(n-1)\mu_{lns}\mathcal{B}_r\mathcal{B}_p y^q C_{jkm} - 2(n-1)\mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkm}$$

$$-2(n-1)\mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkm}$$

$$-2(n-1)\mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q C_{jkm} - 2(n-1)\mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}$$

$$-2(n-1)\mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}$$

$$-2(n-1)\ \mu_l\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q C_{jkm} + \mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C_{jp}^i H_{ik}^p\right)$$

$$+\frac{1}{3}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(R_{jk} - g_{jk}R\right).$$

This shows that

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W_{jk} = a_{lmnsr} W_{jk} + (n-1)b_{lmns}g_{jk} \tag{4.15}$$

$$-2(n-1)\ \mu_{lmns}\mathcal{B}_q y^q C_{jkr}$$

$$-2(n-1)\ \mu_{lmnr}\mathcal{B}_q y^q C_{jks} - 2(n-1)\mu_{lmn}\mathcal{B}_r\mathcal{B}_q y^q C_{jks}$$

$$-2(n-1)\mu_{lmsr}\mathcal{B}_q y^q C_{jkn}$$

$$-2(n-1)\mu_{lms}\mathcal{B}_r\mathcal{B}_q y^q C_{jkn} - 2(n-1)\mu_{lmr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkn}$$

$$-2(n-1)\mu_{lm}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkn}$$

$$-2(n-1)\mu_{lnsr}\mathcal{B}_p y^q C_{jkm} - 2(n-1)\mu_{lns}\mathcal{B}_r\mathcal{B}_p y^q C_{jkm}$$

$$-2(n-1)\mu_{lnr}\mathcal{B}_s\mathcal{B}_q y^q C_{jkm}$$

$$-2(n-1)\mu_{ln}\mathcal{B}_r\mathcal{B}_s\mathcal{B}_q y^q C_{jkm} - 2(n-1)\mu_{lsr}\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}$$

$$-2(n-1)\mu_{ls}\mathcal{B}_r\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}$$

$$-2(n-1)\mu_{lr}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q C_{jkm} - 2(n-1)\ \ \mu_l\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q y^q C_{jkm}$$

If and only if

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(R_{jk} - g_{jk}R\right) = \lambda_{lmnsr}\left(R_{jk} - g_{jk}R\right) \tag{4.16}$$

and

$$\mathcal{B}_r\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C_{jp}^i H_{ik}^p\right) = a_{lmnsr}\left(C_{jp}^i H_{ik}^p\right) \tag{4.17}$$

This following is derived

Theorem 4.4. In G$\mathcal{B}$K -FIRF$_n$, the Ricci Tensor W_{jk} is generalized five recurrent Finsler space if and only if the tensors $\left(C_{jp}^i H_{ik}^p\right)$ and $(R_{jk}–g_{jk}R)$ are five recurrent Finsler space.

CONCLUSION AND RECOMMENDATIONS

The generalized $\mathcal{B}$K -five recurrent space is satisfied in the condition (2.1).

In $G\mathcal{B}K - FIRF_n$, the $\mathcal{B}$-derivative of the five order for Ricci Tensor K_{jk} and curvature vector K_j are given by (2.6) and (2.7), respectively. In $G\mathcal{B}K$ - $FIRF_n$, the necessary and sufficient condition of curvature tensor R_{jkh}^i is generalized five recurrent Finsler space, if and only if the condition $\left(C_{jp}^i H_{hk}^p\right)$ is five recurrent Finsler space and Ricci tensor R_{jk} is generalized five recurrent Finsler space, if and only if the condition $\left(C_{jp}^i H_{ik}^p\right)$ is five recurrent Finsler space. In $G\mathcal{B}K$ - $FIRF_n$, the Wely's projective tensor W_{jkh}^i is generalized five recurrent Finsler space if and only if the tensors $\left(C_{jp}^i H_{hk}^p\right)$ and $\left(\delta_k^i R_{jh} - g_{jk}R_h^i\right)$ are five recurrent Finsler space. In $G\mathcal{B}K$ - $FIRF_n$, the curvature tensor W_{kh}^i is generalized five recurrent Finsler space if and only if the tensor $\left(\delta_k^i H_h - y_k R_h^i\right)$ is five recurrent Finsler space. Finally, Ricci tensor W_{jk} is generalized five recurrent Finsler space if and only if the tensors $\left(C_{jp}^i H_{ik}^p\right)$ and $(R_{jk}–g_{jk}R)$ are five recurrent Finsler space.

The authors call the need for research and study in generalized $\mathcal{B}$K - higher recurrent Finsler spaces and interlard it with the properties of special spaces for Finsler space.

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